Recall $G = SL_2(\mathbb{R}) > P = \begin{pmatrix} * & * \\ & * \end{pmatrix} = MU$, $A = \left\{ \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} : y > 0 \right\}$

$m \in \mathbb{Z}$ "K-type" $\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{=: k_\theta} \mapsto e^{im\theta}$

$$\mathbb{C} \ni s \rightsquigarrow f_s : \Gamma_P U \backslash G \to \mathbb{C}$$

$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} k_\theta \mapsto |y|^{1+s} e(m\theta)$$

$$h_P : G \to \mathbb{C}$$
$$g \mapsto |y|$$

$$\boxed{\begin{aligned} f_s &= h_P^s f_0 \\ |f_s| &= h_P^{1+\sigma}, \quad \sigma = \mathrm{Re}(s) \end{aligned}}$$

$(\mathrm{Re}(s) > 1)$ $E_s : G \to \mathbb{C}$

$$g \mapsto \sum_{\Gamma_P \backslash \Gamma} f_s(\gamma g)$$

Lemma Suppose $\sigma = \mathrm{Re}(s) > 1$, $\sigma \ll 1$.

(i) E_s conv. abs.

(ii) $E_s(g) \ll (\sigma - 1)^{-1} h_P^{1+\sigma}(g)$ $(\Rightarrow E_s :$ mod. growth$)$

(iii) Let $\mathcal{C} :=$ Casimir operator on G $\in Z(\mathfrak{g})$

$= \frac{1}{2} H^2 + EF + FE$, $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$

$F = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}$

Then $\mathcal{C} E_s = \lambda_s E_s$, $\lambda_s = \frac{s^2 - 1}{2}$.

(iv) $E_{s,P} = f_s + c(s) f_{-s}$ for some $c(s) \in \mathbb{C}$

(v) $c(s) \ll (\sigma - 1)^{-1}$. (§1)

Proof

$$|E_s(g)| \le \sum_{\Gamma_P \backslash \Gamma} \underbrace{|f_s(\gamma g)|}_{h_P^{1+\sigma}(\gamma g)} < \infty$$

$$\ll h_P^{1+\sigma}(g) \underbrace{\sum_{\Gamma_P \backslash \Gamma} h_P^{1+\sigma}(\gamma)}$$

$d^\times y = \frac{dy}{y}$

$\delta(\ldots) = y^2$

$$\boxed{E_{s,P}(g) = \int_{\Gamma_U \backslash U} E_s(ug)\,du, \quad h_P(ug) = h_P(g)} \ll \int_0^1 y^{1+\sigma} \underbrace{\frac{d^\times y}{\delta\left(\begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}\right)}}_{\frac{dy}{y^3}} = \frac{1}{\sigma - 1}$$

$\Rightarrow$ (i), (ii).

(iv): last time $\Rightarrow$ (v).

(cf. Harish-Chandra isomorphism)

(iii): $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $\quad E = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ $\quad F = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}$ $\in \mathfrak{g}$

$[E,F] = H \qquad [H,E] = 2E \qquad [H,F] = -2F$

$\Rightarrow \underbrace{FE}_{\in \mathcal{U}(\mathfrak{g})} = EF - [E,F] = EF - H.$

$\Rightarrow \mathcal{C} = \frac{1}{2}H^2 + EF + FE = \frac{1}{2}H^2 - H + 2EF$

Since $\mathcal{C} \in \mathcal{Z}(\mathfrak{g})$, we know a priori that $\mathcal{C} f_s$ satisfies the same transformation properties as f_s:

$$\mathcal{C} f_s \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \, k_\theta \right) = \mathcal{C} f_s(g) \, e^{im\theta}$$

Moreover, $\mathcal{C} f_s$: left-inv. by $\Gamma_\infty = \{ \pm \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \}$.

Any such function is determined by its restriction to A. $\ni a = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$, $y > 0$

$\mathcal{C} f_s(a) = ?$

First, we check that $EF f_s(a) = 0$.

Indeed, $EF f_s(a) = \partial_{t=0} F f_s(a e^{tE})$

$= \partial_{t_1=0} \partial_{t_2=0} f_s(\underbrace{a e^{t_1 E}}_{e^{t_1 \mathrm{Ad}(a) E} a} e^{t_2 F})$

$= \partial_{t_1=t_2=0} f_s(a e^{t_2 F})$

$= 0.$

b/c f_s is left-U-invariant.

$\mathrm{Ad}(a) E = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}^{-1}$

$= \begin{pmatrix} 0 & y^2 \\ & 0 \end{pmatrix} = y^2 E$

$\in \mathrm{Lie}(U)$

$\Rightarrow e^{t_1 \mathrm{Ad}(a) E} \in U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$

$\Rightarrow \mathcal{C} f_s(a) = \left(\frac{1}{2}H^2 - H\right) f_s(a) = \lambda_s f_s(a)$, $\lambda_s = \frac{1}{2}(1+s)^2 - (1+s) = \frac{s^2-1}{2}$

$e^{tH} = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}$ $\qquad a = \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix}$. Set $y = e^t$

$f_s(a) = y^{1+s} = e^{t(1+s)}$

$H \Leftrightarrow \frac{d}{dt}$: acts on $e^{t(1+s)}$ by $1+s$

$\mathcal{C} f_s = \lambda_s f_s \Rightarrow \mathcal{C} E_s = \lambda_s E_s$

defn of E_s, $\mathcal{C} \in \mathcal{Z}(\mathfrak{g})$

any translate of f_s has $\mathcal{C}$-e.v. λ_s.

Lemma Let $f \in C^\infty(\mathbb{R})$ s.t. $\left(\frac{1}{2}\left(\frac{d}{dt}\right)^2 - \frac{d}{dt}\right) f = \lambda_s f$,
$\lambda_s = \frac{s^2-1}{2}$.

Then $\exists\, c_1, c_2 \in \mathbb{C}$ s.t.

$$f(t) = \begin{cases} c_1 e^{t(1+s)} + c_2 e^{t(1-s)} & (s \neq 0) \\ c_1 e^t + c_2 t e^t & (s = 0). \end{cases}$$

$\square$

Lemma Let $\varphi : \Gamma \backslash G \to \mathbb{C}$ be any automorphic form s.t.
(i) φ has K-type m: $\varphi(g k_\theta) = \varphi(g) e^{im\theta}$
(ii) $\mathcal{C}\varphi = \lambda_s \varphi$ $(s \in \mathbb{C})$.
Then $\exists\, c_+, c_- \in \mathbb{C}$ s.t. (for $s \neq 0$, for simplicity)
$\varphi_P = c_+ f_s + c_- f_{-s}$.

Proof φ_P, f_s, f_{-s} are all left $\Gamma_P U$-inv., right K-type m, so it suffices to compare their restrictions to A.

For $a \in A$, we have $\mathcal{C}\varphi_P = \left(\frac{1}{2}H^2 - H\right)\varphi_P$ by the previous proof. View φ_P as a function of t:

$$t \mapsto a = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \mapsto \varphi_P(a).$$

Now apply previous lemma. $\square$

NB Consistent with $E_{s,P} = f_s + c(s) f_{-s}$.
(i.e., $c_+ = 1$, $c_- = c(s)$).

Goal show that E_s and $c(s)$ admit meromorphic continuations to $\mathbb{C}$ that satisfy
(i) $E_{-s} = c(-s) E_s$, (ii) $c(-s)c(s) = 1$.

NB $E_{-s,P} = f_{-s} + c(-s) f_s$, $(c(-s)E_s)_P = c(-s) f_s + c(-s)c(s) f_{-s}$
$E_{s,P} = f_s + c(s) f_{-s}$, so (ii) $\Leftarrow$ (i) holds after taking constant terms along P.

Defn $I_c^\infty(G) := \{ \alpha \in C_c^\infty(G) : \alpha(kxk^{-1}) = \alpha(x) \ \forall\, x \in G, k \in K \}$

NB if $\varphi : G \to \mathbb{C}$ has right K-type m,
then so does $\varphi * \alpha \quad \forall\, \alpha \in I_c^\infty(G)$.

Lem $\forall\, s \in \mathbb{C}$, $\forall\, \alpha \in I_c^\infty(G)$, $\exists\, \hat{\alpha}(s)$ s.t.
$\forall\, \varphi \in C^\infty(G)$ with $\mathcal{C}\varphi = \lambda_s \varphi$, right K-type m,
we have $\varphi * \alpha = \hat{\alpha}(s)\, \varphi$.

Proof more precise form of the result of H-C (cf. Ex 3.8).

Example $\varphi := f_s$. Then $f_s(1) = 1$, so

entire in s ; compactly-supported in G

$$\hat{\alpha}(s) = (f_s * \alpha)(1) = \int_G f_s(g)\, \alpha(g^{-1})\, dg$$

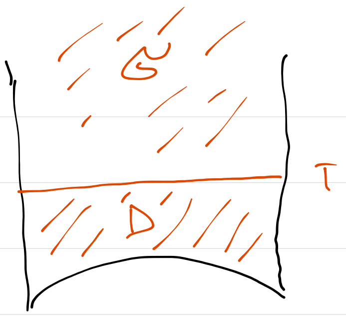
$$= \int_{x,y,\theta} (\cdots).$$

$\Rightarrow \hat{\alpha}$ is entire in s, inv. under $s \mapsto -s$.

Remark Let $\alpha_n \in I_c^\infty(G)$ be any "approximate identity":
$\mathrm{supp}(\alpha_n) \to \{1\}$, $\int_G \alpha_n = 1$.

Then for fixed s, we have $\hat{\alpha}_n(s) \longrightarrow 1$ as $n \to \infty$.
More generally, this holds uniformly for $s \in$ fixed bounded set.
In particular, $\forall$ bounded $U \subseteq \mathbb{C}$, $\varepsilon > 0$ $\exists\, \alpha \in I_c^\infty(G)$ s.t.
$|\hat{\alpha}(s) - 1| \le \varepsilon \quad \forall\, s \in U$.
$\Rightarrow |\hat{\alpha}(s)| \ge 1 - \varepsilon \ge 1/2$ if $\varepsilon \le 1/2$.

$\Gamma \backslash G$



$\Gamma_U \backslash U\mathfrak{S} \underset{\rho}{\simeq} \mathfrak{S}$
up to measure zero

Fix a Siegel domain $\mathfrak{S} = \mathfrak{S}_T$ with $T > \frac{\sqrt{3}}{2}$ large enough that $\Gamma_U \backslash U\mathfrak{S} \hookrightarrow \Gamma \backslash G$.

D: the compact subset of $\Gamma \backslash G$ complementary to $\mathfrak{S}$.

$$H_D := \left\{ \varphi \in L^2 := L^2(\Gamma \backslash G) \text{ of right } K\text{-type } m : \varphi_P|_{\mathfrak{S}} = 0 \right\} \underset{\text{closed subspace}}{\subseteq} L^2$$

For $\varphi \in L^2(\Gamma \backslash G)$ of K-type m, we may write

$$\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} k_\theta \right) = \sum_{\ell \in \mathbb{Z}} e(\ell x)\, \varphi_\ell(y)\, e^{im\theta},$$

$$\varphi_\ell(y) = \int_{\mathbb{R}/\mathbb{Z}} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \right) e(-\ell x)\, dx.$$

Then $\varphi_P \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} = \varphi_0(y)$.

Thus $\varphi \in H_D \iff \varphi_0(y) = 0$ for $\begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \in \mathfrak{S}$. $(y^2 > T)$

Write $\Lambda : L^2 \to H_D$ for the orthogonal projection. (Called "truncation" operator.)

Exercise $\Lambda \varphi \overset{(*)}{=} \varphi - \chi \varphi_P$, $\chi :=$ characteristic function of $\mathfrak{S}$.

(*) on the fundamental domain pictured above

Lemma Let $\alpha \in I_c^\infty(G)$.
Write $(\star\alpha) : \varphi \mapsto \varphi \star \alpha$.
The following operators are compact:

(i) $(\star\alpha) : H_D \longrightarrow L^2$

(ii) $(\star\alpha) \circ \Lambda : L^2 \longrightarrow L^2$

(iii) $\Lambda \circ (\star\alpha) : H_D \longrightarrow H_D$.

Proof (i) another variant on "approximation by constant terms":
$\forall \varphi \in H_D$, $\varphi \star \alpha$ decays rapidly,
hence $(\star\alpha) : H_D \to L^\infty$ is bounded,
and similarly for derivatives.
(ii) $\Leftarrow$ (i) (defn of Λ)
(iii) $\Leftarrow$ (i) (Λ : continuous)

Lemma Let $\varphi : \Gamma\backslash G \to \mathbb{C}$ be locally integrable,
of right K-type m, and of moderate growth.
Then $\Lambda(\varphi \star \alpha) \in H_D$, where Λ is defined by the formula $(\star)$.
Proof Just need to check that $\Lambda(\varphi \star \alpha) \in L^2$.
Indeed, note first that $\varphi \star \alpha$: smooth, uniform moderate growth, hence by "approximation by constant term", $\varphi \star \alpha$ is approximated by $(\varphi \star \alpha)_P$ near ∞.

$\Longrightarrow$ $\varphi \star \alpha - (\varphi \star \alpha)_P$ decays near ∞, hence is L^2 near ∞.

$\Longrightarrow$ $\Lambda(\varphi \star \alpha) \in H_D$. $\square$

$\varphi \star \alpha$: smooth